6

Three-Dimensional Flow and Heat Transfer within Highly Anisotropic Porous Media

Numerical Determination of Permeability Tensor, Inertial Tensor, and Interfacial Heat Transfer Coefficient

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6.1 Introduction

In order to design efficient heat transfer equipment, one must know the details of both flow and temperature fields within the equipment. Such detailed flow and temperature fields within a manmade assembly may be investigated numerically by solving the set of governing equations based on the first
principles (i.e., continuity, momentum, and energy balance equations), so as to resolve all scales of flow and heat transfer in the system. However, in reality, it would hardly be possible to reveal such details even with the most powerful supercomputer available today. For example, a grid system, designed for a comparatively large scale of heat exchanger systems, would not be fine enough to describe the details of flow and heat transfer around a fin in a heat transfer element.

It has been recently pointed out by DesJardin (personal communication, 2001) and many others [1,2] that the concept of local volume-averaging theory, namely, VAT, widely used in the study of porous media [3–5] may be exploited to investigate the flow and heat transfer within such a complex heat and fluid flow equipment. These complex assemblies usually consist of small-scale elements, such as a bundle of tubes and fins, which one does not want to grid. Under such a difficult situation, one may resort to the concept of VAT instead, so as to establish a macroscopic model, in which these collections of small-scale elements are treated as highly anisotropic porous media. There are a number of situations in which one has to introduce macroscopic models to describe complex fluid flow and heat transfer systems.

Nakayama and Kuwahara [6] appealed to VAT and derived a set of macroscopic governing equations for turbulent heat and fluid flow through an isotropic porous medium in local thermal equilibrium. The resulting set of governing equations was generalized by Nakayama et al. [7], to treat highly anisotropic porous media by integrating the microscopic governing equations, namely, the Reynolds averaged versions of continuity, Navier-Stokes, and energy equations. One can conveniently use these macroscopic equations designed for highly anisotropic porous media, to investigate the flow and heat transfer within complex equipment, since a single set of the volume-averaged governing equations can be applied to the entire calculation domain within the complex heat transfer equipment consisting of both large- and small-scale elements. All that one has to do is to specify the spatial distributions of macroscopic model parameters such as porosity and permeability. The clear fluid flow region without small-scale obstructions, for example, will be treated as a special case, as one sets the porosity for unity with an infinitely large permeability.

In order to utilize these macroscopic equations for such large-scale numerical computations, one must close the macroscopic equations by modeling the flow resistance associated with individual subscale solid elements and also the heat transfer rate between the flowing fluid and the subscale elements, in terms of the macroscopic velocity vector and relevant geometrical parameters. Such subscale models can be established by carrying out direct numerical experiments at a pore scale for individual subscale elements. Since the subscale structure is often periodic, the numerical experiment can be performed economically, focusing on one structural unit and utilizing periodic boundary conditions there. The microscopic results, thus obtained, are processed to extract the macroscopic hydrodynamic and
thermal characteristics, and eventually to determine the unknown model
constants of the subscale models associated with permeability tensor, inertial
(Forchheimer) tensor, and interfacial heat transfer coefficient. Kuwahara
et al. [8], Nakayama and Kuwahara [9], Nakayama et al. [10], and
De Lemos and Pedras [11,12] have conducted such microscopic computations
successfully. The unknown model constants including the interfacial heat
transfer coefficient, permeability, and Forchheimer constants were deter-
mined by carrying out exhaustive numerical experiments using a periodic
array of square and circular cylinders. A review on the research towards
this endeavor may be found in chapter 10 of the first edition of the
handbook [13].

All these investigations, however, were limited to the cases of the cross-
flows over two-dimensional structures. In reality, all manmade elements such
as those in plate fin heat exchangers are three-dimensional in nature. Natu-
really, the macroscopic velocity vector is not always perpendicular to the axis of
the cylinder. The deviating angle between the velocity vector and the plane
perpendicular to the axis of the cylinder is called “yaw” angle. Thus, the
three-dimensional yaw effects on the permeability tensor, inertial tensor, and
interfacial heat transfer coefficient must be elucidated beforehand, in order
to design such heat transfer elements and systems. Nakayama et al. [14]
used a bundle of rectangular cylinders to describe such three-dimensional
anisotropic porous media, and showed that, under macroscopically uniform
flow, the three-dimensional governing equations reduce to quasi-three-
dimensional forms, in which all derivatives associated with the axis of the
cylinder can be either eliminated or replaced by other determinable expres-
sions. Thus, only two-dimensional storages are required for the dependent
variables. This quasi-three-dimensional numerical calculation procedure has
been exploited to investigate the three-dimensional effects on the permea-
bility tensor, inertial tensor, and interfacial heat transfer coefficient, which are
needed to close the proposed set of the macroscopic governing equations.

In what follows, we shall review a series of extensive investigations on
three-dimensional flow and heat transfer within highly anisotropic porous
media. A bank of long cylinders is considered as one of fundamental geo-
metrical configurations often found in heat exchangers and many other
manmade anisotropic porous media. Numerical determination of the impor-
tant subscale model parameters, such as permeability tensor, inertial tensor,
and interfacial heat transfer coefficient, will be described in detail, so as
to elucidate the three-dimensional yaw effects on these macroscopic hydro-
dynamic and thermal parameters. The results are compared with available
experimental data to substantiate the validity of the present modeling strategy
for three-dimensional flow and heat transfer within highly anisotropic por-
ous media. Upon correlating these macroscopic results, a useful set of
explicit expressions will be established for the permeability tensor, inertial
Forchheimer tensor, and interfacial heat transfer coefficient, so as to charac-
terize three-dimensional flow and heat transfer through a bank of infinitely
long cylinders in yaw.
6.2 Volume-Averaged Governing Equations

According to Nakayama et al. [2,15], the set of the macroscopic equations based on VAT for the case of laminar flow through an anisotropic porous medium runs as

\[
\frac{\partial \langle u_j \rangle^f}{\partial x_j} = 0
\]  
(6.1)

\[
\rho_t \left( \frac{\partial \langle u_i \rangle^f}{\partial t} + \frac{\partial}{\partial x_j} \langle u_j \rangle^f \langle u_i \rangle^f \right) = -\frac{\partial \langle p \rangle^f}{\partial x_i} + \frac{\partial}{\partial x_i} \left[ \mu_t \left( \frac{\partial \langle u_j \rangle^f}{\partial x_j} + \frac{\partial \langle u_j \rangle^f}{\partial x_i} \right) \right] \\
- \phi \left( \mu_t K_{ij}^{-1} + \phi \rho_t b_{ijf} \left( \langle u_k \rangle^f \langle u_k \rangle^f \right)^{1/2} \right) \langle u_j \rangle^f
\]  
(6.2)

\[
\rho_c \rho_t \left( \frac{\partial \langle T \rangle^f}{\partial t} + \frac{\partial}{\partial x_j} \langle u_j \rangle^f \langle T \rangle^f \right) = \frac{\partial}{\partial x_j} \left[ \frac{k_t}{c_p} \frac{\partial \langle T \rangle^f}{\partial x_j} + \frac{1}{\rho_t \gamma} \int_{A_{nt}} k_t \langle T \rangle^f \, dA - \rho_t c_p \langle u_j \rangle^f \langle T \rangle^f \right] \\
+ h_t a_i (\langle T \rangle^s - \langle T \rangle^f)
\]  
(6.3)

where

\[
a = \langle a \rangle^f + a'
\]  
(6.4a)

and

\[
\langle a \rangle^f = \frac{1}{V_t} \int_{V_t} a \, dV
\]  
(6.4b)

in general denotes the intrinsic averaged value of \( a \) over the volume space \( V_t \) occupied by the fluid, whereas \( a' \) denotes its spatial deviation. In fact, the idea of VAT is quite near to that of the representative elementary volume. However, the size of the elementary volume \( V \) should be large enough to cover the microscopic structure, but, at the same time, much smaller than the macroscopic scale. The sub- and superscripts \( f \) and \( s \) stand for the fluid and solid phases, respectively. In the foregoing momentum and energy equations, the terms associated with the microscopic structure are modeled according to

\[
\frac{1}{V_t} \int_{A_{nt}} \mu_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j \, dA - \rho_t \frac{\partial}{\partial x_j} \langle u_j \rangle^f
\]

\[
= -\phi \left( \mu_t K_{ij}^{-1} + \phi \rho_t b_{ijf} \left( \langle u_k \rangle^f \langle u_k \rangle^f \right)^{1/2} \right) \langle u_j \rangle^f
\]  
(6.5)
and

\[
\frac{1}{V} \int_{A_{\text{int}}} k_{ij} \frac{\partial T}{\partial x_j} n_j \, dA = h_{ij} a_f (\langle T \rangle^f - \langle T \rangle^s)
\]

(6.6)

where \( \phi = V_f / V \) is the porosity, and \( n_j \) is the unit vector normal to the interface pointing from the fluid side to solid side. Equation (6.5) is a generalized form of Forchheimer-extended-Darcy’s law. The net heat transfer between the fluid and solid is given by \( h_{ij} a_f (\langle T \rangle^f - \langle T \rangle^s) \) upon introducing the interfacial heat transfer coefficient \( h_f \), where \( a_f = A_{\text{int}} / V \) is the specific interfacial area.

In order to close the foregoing set of the macroscopic governing equations, we must determine the permeability tensor \( K_{ij} \) and Forchheimer tensor \( h_{ij} \) appearing in Eq. (6.5) and also the interfacial heat transfer coefficient \( h_f \) appearing in Eq. (6.6), for a given microscopic structure. As will be demonstrated later, such subscale models can be established by conducting microscopic numerical experiments for individual subscale elements. Then, the microscopic results are fed into the LHS terms of Eqs. (6.5) and (6.6) to determine these unknown tensors and coefficient as functions of the macroscopic quantities. When the structure is geometrically periodic, only one structural unit may be taken as a calculation domain.

6.3 Preliminary Consideration of Macroscopically Uniform Flow Through an Isothermal Porous Medium

In order to appreciate the foregoing macroscopic governing equations, we consider one of the most fundamental flows through a manmade structure, namely, a macroscopically uniform steady flow through an isothermal three-dimensional periodic structure of infinite extent as shown in Figure 6.1. The body shape of the structural element is arbitrary, and its arrangement can be aligned as in Figure 6.1 or staggered in an arbitrary fashion. Let us find the macroscopic pressure and temperature solutions using the foregoing macroscopic momentum and energy equations.

Upon referring to the orthogonal unit vectors \((\vec{l}, \vec{m}, \vec{n})\) as shown in Figure 6.1, the macroscopically steady and uniform velocity field may be presented by

\[
\langle \vec{u} \rangle = |\langle \vec{u} \rangle| (\cos \alpha \vec{l} + \cos \beta \vec{m} + \cos \gamma \vec{n})
\]

(6.7)

where

\[
\langle \vec{u} \rangle = \frac{1}{V} \int_V \vec{u} \, dV
\]

(6.8)

is the Darcian velocity, which differs from the intrinsic average velocity (given by Eq. 6.4[b]) by the factor \( \phi \), such that \( \langle \vec{u} \rangle = \phi \langle \vec{u} \rangle^f \). The local volume \( V \) for
integration may be taken as the structural volume element as indicated by dashed lines in the figure. The directional cosines of the volume-averaged macroscopic velocity vector satisfy the obvious relationship, namely,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$  \hspace{1cm} (6.9)

This relation may be rewritten equivalently using the cross-flow angle $\alpha'$ projected onto the $x-y$ plane as

$$\cos \alpha = \sin \gamma \cos \alpha' \quad \text{and} \quad \cos \beta = \sin \gamma \sin \alpha'$$  \hspace{1cm} (6.10)

Under the macroscopically uniform velocity as given by Eq. (6.7), the volume-averaged momentum equation (6.2) reduces to

$$-\frac{\partial \langle p \rangle}{\partial x_i} = (\mu_i K_{ij}^{-1} + \rho_i b_{ij}(\bar{u}_j))(u_j)$$  \hspace{1cm} (6.11)

where

$$\langle u_k \rangle \langle u_k \rangle = |(\bar{u})|^2$$  \hspace{1cm} (6.12)

Thus, the macroscopic momentum equation leads to the Forchheimer extended Darcy's law [16], generalized for the case of anisotropic porous media.

We shall assume that the wall surfaces of the structure are maintained at a constant temperature. Then, the microscopic temperature field, when
averaged spatially within a local structural control volume \( V \), should lead to the macroscopic temperature field whose gradient aligns with the macroscopic velocity vector in the \( s \) direction, such that the volume-averaged energy equation (6.3), under the macroscopically steady and uniform velocity field with negligible macroscopic longitudinal conduction reduces to

\[
\rho_f c_p f \frac{d\langle T \rangle^f}{ds} = -h_f a_f ((T)^f - \langle T \rangle^s) \tag{6.13}
\]

where

\[
ds = \cos \alpha \, dx + \cos \beta \, dy + \cos \gamma \, dz \tag{6.14}
\]

Since the surface temperature of the structure \( \langle T \rangle^s \) is constant, Eq. (6.13) naturally yields the macroscopic temperature field as

\[
(T)^f - \langle T \rangle^s = \langle T \rangle^f - \langle T \rangle^s \rangle_{\text{ref}} \exp \left( -\frac{a_f h_f}{\rho_f c_p f (\bar{u})} (s - s_{\text{ref}}) \right) \tag{6.15}
\]

Note that the interfacial heat transfer coefficient \( h_f \) is expected to be constant for the periodically fully developed heat and fluid flow, as in the cases of thermally fully developed tube and channel flows. The correct set of the periodic boundary conditions should lead to the microscopic temperature field compatible with the macroscopic temperature field as given by Eq. (6.15). (In other words, the resulting microscopic temperature field, when averaged spatially, must yield the macroscopic temperature field given by Eq. [6.15].)

### 6.4 Periodic Boundary Conditions for Three-Dimensional Periodic Structure

The periodic boundary conditions needed to conduct microscopic numerical experiments for manmade structures must be compatible with the foregoing macroscopic solutions for the macroscopically uniform flow. The prescription of the periodic boundary conditions for the velocity field (or pressure field instead) is rather straightforward, since the profiles at both upstream and downstream boundaries must be identical. Patankar et al. [17] prescribed the pressure drop over one structural unit to attack the problem of fully developed flow and heat transfer in ducts having streamwise-periodic variations of cross-sectional area, while Nakayama et al. [18] and Kuwahara et al. [19] chose to prescribe the mass flow rate (rather than the pressure drop) to obtain the fully developed velocity and temperature fields within two-dimensional periodic arrays. However, the prescription of the periodic temperature field requires some consideration, when the surface wall temperature is kept constant. Naturally, the temperature difference between the fluid and solid wall
becomes vanishingly small at the fully developed stage, as in the case of thermally fully developed tube flow with uniform surface temperature.

In what follows, we shall seek an appropriate set of the periodic boundary conditions to impose along such periodic boundaries of the structure. Let us consider one of the simplest temperature fields, namely, the fully developed temperature field for the case of forced convection from isothermal parallel plates with a channel height $H$, as shown in Figure 6.2.

The thermally fully developed flow of this kind may be regarded as one of the special periodically fully developed flows, since the temperature profile at $x = x_0$ is similar to that at $x = x_0 + L$, such that

$$\frac{T(x_0 + L, y) - T_w}{T_B(x_0 + L) - T_w} = \frac{T(x_0, y) - T_w}{T_B(x_0) - T_w} \quad (6.16)$$

where $L$ is any axial distance of arbitrary size (which may be unlimitedly large or small), and $T_B$ is the bulk mean temperature. This can be rearranged as

$$\frac{T(x_0 + L, y) - T_w}{T(x_0, y) - T_w} = \frac{T_B(x_0 + L) - T_w}{T_B(x_0) - T_w} = \exp \left(-\frac{2h_f L}{\rho f c_p u_B H} \right) \quad (6.17)$$

where $u_B$ is the bulk mean velocity. The last expression in the RHS comes from the macroscopic temperature solution given by Eq. (6.15), as we note that $|\langle \bar{u} \rangle| = u_B, \langle T \rangle^f = T_B, \langle T \rangle^b = T_w$, and $a_i = 2/H$ for this case. Selecting a reference axial distance $L_0$ along an arbitrary level at $y = y_0$ gives

$$\frac{T(x_0 + L_0, y_0) - T_w}{T(x_0, y_0) - T_w} = \exp \left(-\frac{2h_f L_0}{\rho f c_p u_B H} \right) \quad (6.18)$$

Upon combining Eqs. (6.17) and (6.18), we obtain

$$T(x_0 + L, y) - T_w = (T(x_0, y) - T_w) \exp^{L/L_0} \quad (6.19)$$
where

\[ \tau \equiv \frac{T(x_0 + L_0, y_0) - T_w}{T(x_0, y_0) - T_w} \]  

(H6.20)

Hence, Eq. (6.19) is one of the many possible expressions for the thermally periodic boundary condition for this simple case, which guarantees us to provide the microscopic temperature field compatible with the macroscopic temperature field as given by Eq. (6.15). It is straightforward to extend the case to an infinite series of flat plates of finite length, to the two-dimensional periodic structure of arbitrary shape, and finally to a general three-dimensional periodic structure, as shown in Figure 6.1, as done by Nakayama et al. [14].

Thus, the steady-state microscopic governing equations and their correct set of the boundary conditions for periodically fully developed heat and fluid flow through a three-dimensional periodic structure are given as follows:

\[ \nabla \cdot \vec{u} = 0 \]  

(H6.21a)

\[ \rho_l (\nabla \cdot \vec{u}) = -\nabla p + \mu_l \nabla^2 \vec{u} \]  

(H6.21b)

\[ \rho_l c_p T \nabla \cdot (\vec{u} T) = k_l \nabla^2 T \]  

(H6.21c)

On the solid walls:

\[ \vec{u} = \vec{0} \]  

(H6.24a)

\[ T = T_w (= T^*) \]  

(H6.24b)

On the periodic boundaries:

\[ \vec{u} \big|_{x=-L/2} = \vec{u} \big|_{x=L/2} \]  

(H6.25a)

\[ \vec{u} \big|_{y=-H/2} = \vec{u} \big|_{y=-H/2} \]  

(H6.25b)

\[ \vec{u} \big|_{z=-M/2} = \vec{u} \big|_{z=-M/2} \]  

(H6.25c)

where the origin of the Cartesian coordinates \((x, y, z)\) is set in the center of the structural unit \((-L/2 \leq x \leq L/2, -H/2 \leq y \leq H/2, -M/2 \leq z \leq M/2)\), as indicated in Figure 6.1. The mass flow rate constraints based on Eq. (6.7) are given by:

\[ \int_{-M/2}^{M/2} \int_{-H/2}^{H/2} u \, dy \, dz \big|_{x=\pm L/2} = \int_{-M/2}^{M/2} \int_{-L/2}^{L/2} u \, dy \, dz \big|_{x=\pm L/2} = HM \cos \alpha (|\vec{u}|) \]  

(H6.26a)

\[ \int_{-M/2}^{M/2} \int_{-L/2}^{L/2} v \, dx \, dz \big|_{y=\pm H/2} = \int_{-M/2}^{M/2} \int_{-L/2}^{L/2} v \, dx \, dz \big|_{y=\pm H/2} = LM \cos \beta (|\vec{u}|) \]  

(H6.26b)
\[ \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} w \, dx \, dy \bigg|_{z=-M/2} = \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} w \, dy \, dx \bigg|_{z=M/2} = LH \cos \gamma (|\vec{u}|) \]  

(6.26c)

Finally, the thermal boundary conditions for the periodic boundaries are given by

\[ (T - T_w)_{x=L/2} = \tau (L \cos \alpha)/(L \cos \alpha + H \cos \beta + M \cos \gamma) \quad (T - T_w)_{x=-L/2} \]  

(6.27a)

\[ (T - T_w)_{y=H/2} = \tau (H \cos \beta)/(L \cos \alpha + H \cos \beta + M \cos \gamma) \quad (T - T_w)_{y=-H/2} \]  

(6.27b)

\[ (T - T_w)_{z=M/2} = \tau (M \cos \gamma)/(L \cos \alpha + H \cos \beta + M \cos \gamma) \quad (T - T_w)_{z=-M/2} \]  

(6.27c)

where

\[ \tau = \frac{(T - T_w)_{x=L/2, y=H/2, z=M/2}}{(T - T_w)_{x=-L/2, y=-H/2, z=-M/2}} \]  

(6.28)

The literature survey [14] has revealed that no explicit periodic thermal boundary conditions (such as given by Eqs. [6.27]) have been reported before for three-dimensional periodic heat and fluid flows of this kind.

6.5 Quasi-Three-Dimensional Numerical Calculation Procedure

The foregoing set of governing equations and corresponding boundary conditions may greatly be simplified for the case of the three-dimensional heat and fluid flow through a two-dimensional periodic structure such as a bank of cylinders in yaw, as illustrated in Figure 6.3(a) and more specifically in Figure 6.3(b) to show the cross-sectional plane of the square cylinder bank subject to the present numerical experiment. All square cylinders in the figure, which may be regarded as heat sinks (or sources), are maintained at a constant temperature \( T_w (=T^*) \), which is lower (or higher) than the temperature of the flowing fluid. Since the cylinders are infinitely long, the set of the governing equations (6.21) to (6.23) reduces to a quasi-three-dimensional form, in consideration of the limiting case, namely, \( M \to 0 \):

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

(6.29)

\[ \frac{\partial}{\partial x} \left( u^2 - v \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( vu - v \frac{\partial u}{\partial y} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} \]  

(6.30)
FIGURE 6.3
Two-dimensional periodic structure; (a) bank of circular cylinders, (b) bank of square cylinders (cross-sectional view).

\[
\frac{\partial}{\partial x} \left( \mu \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left( \nu \frac{\partial v}{\partial y} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial y} \tag{6.31}
\]

\[
\frac{\partial}{\partial x} \left( \mu \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( \nu \frac{\partial w}{\partial y} \right) = \frac{v}{A_{\text{fluid}}} \int_{P_{\text{int}}} \frac{\partial w}{\partial n} dP \tag{6.32}
\]

\[
\frac{\partial}{\partial x} \left( uT - \frac{v}{P_{rt}} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( vT - \frac{v}{P_{rt}} \frac{\partial T}{\partial y} \right) = S_w \tag{6.33}
\]

where \( P \) is the coordinate along the wetted periphery, whereas \( n \) is the coordinate normal to \( P \) pointing inward from the peripheral wall to fluid side. \( A_{\text{fluid}} \) is the passage area of the fluid, and

\[
S_w = -\frac{\partial}{\partial z} \left( wT - \frac{v}{P_{rt}} \frac{\partial T}{\partial z} \right) = -\left( w - \frac{v}{P_{rt}} \frac{\cos \gamma \ln \tau_0}{L \cos \alpha + H \cos \beta} \right) \frac{\cos \gamma \ln \tau_0}{L \cos \alpha + H \cos \beta} (T - T_w)_{z=0} \tag{6.34}
\]
since
\[
\frac{\partial T}{\partial z} = (T - T_w)_{|z=0} \lim_{M \to 0} \frac{\tau(M \cos \gamma)/(L \cos \alpha + H \cos \beta + M \cos \gamma) - 1}{M} = \frac{(T(x, y, 0) - T_w) \cos \gamma}{L \cos \alpha + H \cos \beta} \ln \tau_0
\]

(6.35)

where
\[
\tau_0 \equiv \tau|_{z=0} = \frac{(T - T_w)|_{x=L/2,y=H/2,z=0}}{(T - T_w)|_{x=-L/2,y=-H/2,z=0}}
\]

(6.36)

The boundary and compatibility conditions for the periodic planes are given by

\[
\tilde{u} \bigg|_{x=-L/2} = \tilde{u} \bigg|_{x=L/2}
\]

(6.37a)

\[
\tilde{v} \bigg|_{y=-H/2} = \tilde{v} \bigg|_{y=H/2}
\]

(6.37b)

\[
\int_{-H/2}^{H/2} u \, dy \bigg|_{x=-L/2} = \int_{-H/2}^{H/2} u \, dy \bigg|_{x=L/2} = H \cos \alpha \langle \tilde{u} \rangle
\]

(6.38a)

\[
\int_{-L/2}^{L/2} v \, dx \bigg|_{y=-H/2} = \int_{-L/2}^{L/2} v \, dx \bigg|_{y=H/2} = L \cos \beta \langle \tilde{u} \rangle
\]

(6.38b)

\[
\int_{-H/2}^{H/2} \int_{-L/2}^{L/2} \omega \, dx \, dy = LH \cos \gamma \langle \tilde{u} \rangle
\]

(6.38c)

\[
(T - T_w)|_{x=L/2} = \tau_0 \frac{(L \cos \alpha)/(L \cos \alpha + H \cos \beta)}{(T - T_w)|_{x=-L/2}}
\]

(6.39a)

\[
(T - T_w)|_{y=-H/2} = \tau_0 \frac{(H \cos \beta)/(L \cos \alpha + H \cos \beta)}{(T - T_w)|_{y=H/2}}
\]

(6.39b)

In this way, all derivatives associated with z can be eliminated. Thus, only two-dimensional storages are required to solve Eqs. (6.29) to (6.33). (Note that both Eqs. (6.32) and (6.33) may be treated as two-dimensional scalar transport equation.)

6.6 Method of Computation and Preliminary Numerical Consideration

The governing equations (6.29) to (6.31) subject to the foregoing boundary and compatibility conditions (6.37a), (6.37b), (6.38a), and (6.38b) were numerically
solved using SIMPLE algorithm proposed by Patankar and Spalding [20]. As the $u$ and $v$ velocity fields were established, the remaining equations (6.32) and (6.33) subject to the boundary conditions (6.37c), (6.38c), (6.39a), and (6.39b) were solved to find $w$ and $T$. Convergence was measured in terms of the maximum change in each variable during an iteration. The maximum change allowed for the convergence check was set to $10^{-5}$, as the variables are normalized by appropriate references. The hybrid scheme has been adopted for the advection terms. Further details on this numerical procedure can be found in Patankar [21] and Nakayama et al. [22]. For the cases of square cylinder banks, all computations have been carried out for a one structural unit $L \times H$, as indicated by dashed lines in Figure 6.3(b), using nonuniform grid arrangements with $91 \times 91$, after comparing the results against those obtained with $181 \times 181$ for some selected cases, and confirming that the results are independent of the grid system. All computations were performed using the computer system at Shizuoka University Computer Center.

In order to confirm the validity of the present numerical procedure based on the periodic boundary conditions, preliminary computations were also conducted for the case of forced convection from isothermal parallel plates with a channel height $H$, as shown in Figure 6.2. Since $\alpha = 0$, $\beta = \gamma = \pi/2$ for this case, we find $w = S_w = 0$, and

$$
Nu_{2H} = \frac{h_f(2H)}{k_f} = \frac{\rho c_p u_B H^2}{L k_f} \ln \left( \frac{1}{\tau_0} \right)
$$

(6.40)

from Eqs. (6.18) and (6.39a). The computations were made for $10 \leq Re_{2H} \leq 10^3$ and $Pr = 1$, and the numerical results for $Nu_{2H}$ are presented in Figure 6.4. The predicted Nusselt number attains its fully developed value, namely, $Nu_{2H} = 7.54$, which coincides with the exact solution.

![Figure 6.4](image)

Fully developed Nusselt number in a channel.
6.7 Validation of Quasi-Three-Dimensional Calculation Procedure

The efficiency and accuracy of the quasi-three-dimensional calculation procedure, proposed for the two-dimensional structure, may be examined by comparing the results based on the procedure with those based on the full three-dimensional calculation procedure. Extensive calculations have been carried out using the full three-dimensional governing equations (6.21) to (6.23) for macroscopically uniform flow through a bank of square cylinders in yaw, as illustrated in Figure 6.3(b).

Computations may be made using the dimensionless equations based on the absolute value of the Darcian velocity vector $|\langle \vec{u} \rangle|$, and the longitudinal center-to-center distance $L$ as reference scales. For carrying out a series of numerical calculations, it may be convenient to use the Reynolds number based on $L$ as $Re_L = |\langle \vec{u} \rangle|L/\nu_l$, which can readily be translated into the Reynolds number based on the size of square rod $D$ as follows:

$$Re_D = |\langle \vec{u} \rangle|D/\nu_l = \left( \frac{1 - \phi}{L} \right)^{1/2} Re_L$$

(6.41)

where the porosity is given by

$$\phi = 1 - \left( \frac{D^2}{HL} \right)$$

(6.42)

In this numerical experiment, the Reynolds number is varied from $10^{-2}$ to $6 \times 10^3$, as in the study for the cross-flows (i.e., with $\gamma = \pi/2$)[10]. For this time, both cross-flow angle $\alpha'$ and yaw angle $\gamma$ are varied from 0 to $\pi/2$ with an increment $\pi/36$ to cover all possible macroscopic flow directions in the three-dimensional space, such that entire solution surfaces may be constructed over the domain $0 \leq \alpha' \leq \pi/2$ and $0 \leq \gamma \leq \pi/2$. Moreover, the ratio $H/L$ is set to 1, $\frac{3}{2}$, and 2 to investigate the effects of the degree of the anisotropy, whereas the ratio $D/L$ is fixed to $\frac{1}{2}$ for all calculations.

In Figures 6.5, the resulting velocity and temperature fields obtained for the case of $H/L = 1, \alpha' = 45^\circ, \gamma = 45^\circ, Re_L = 600,$ and $Pr = 1$ using the full three-dimensional calculation procedure (Figure 6.5[a]) are compared with those based on the quasi-three-dimensional calculation procedure based on the simplified governing equations (6.29) to (6.33) (Figure 6.5[b]). Excellent agreement between the two sets of the results can be seen, which verifies the accuracy and efficiency of the proposed quasi-three-dimensional calculation procedure. The CPU time required for the convergence using the full three-dimensional calculation turned out to be roughly 3 h, 6 times more than that using the quasi-three-dimensional calculation. This proves the effectiveness of the quasi-three-dimensional calculation procedure.
This economical quasi-three-dimensional calculation procedure has been used to conduct a numerical experiment for macroscopically uniform flow through a bank of square cylinders in yaw over a wide range of the Reynolds number and flow angle.

### 6.8 Determination of Permeability Tensor

The gradient of the intrinsic average pressure may readily be evaluated using the microscopic results as

\[
- \frac{\partial p^f}{\partial s} = \frac{\cos \alpha}{L(H-D)} \int_{-(H-D)/2}^{(H-D)/2} (p|_{z=-L/2} - p|_{z=L/2}) dy + \frac{\cos \beta}{H(L-D)} \int_{-(L-D)/2}^{(L-D)/2} (p|_{y=-H/2} - p|_{y=H/2}) dy + \frac{\mu \cos \gamma}{(HL-D^2)} \oint_{P_1} \frac{\partial w}{\partial n} dP
\]

(6.43)
When the velocity (i.e., Reynolds number) is low, the proposed model equation (6.11) reduces to Darcy's law as

\[ -\frac{\partial (p)^f}{\partial x_i} = \left( \mu_T K_{i_j}^{-1} + \rho_T \beta_{i_j} \right) (u_j) \approx \mu_T K_{i_j}^{-1} (u_j) \]  \hspace{1cm} (6.44)

For the orthotropic media, the permeability tensor may be modeled following Dullien [23] as

\[ K_{i_j}^{-1} = \left( l_i l_j / K_{i_1} + (m_i m_j) / K_{i_2} + (n_i n_j) / K_{i_3} \right) \]  \hspace{1cm} (6.45)

such that

\[ -\frac{\partial (p)^f}{\partial x_i} \approx \mu_T K_{i_j}^{-1} (u_j) = \left( \frac{\cos \alpha}{K_{i_1}} l_i + \frac{\cos \beta}{K_{i_2}} m_i + \frac{\cos \gamma}{K_{i_3}} n_i \right) \frac{1}{|\langle \vec{u} \rangle|} \]  \hspace{1cm} (6.46)

where

\[ \cos \alpha = \frac{l_j (u_j)}{|\langle \vec{u} \rangle|}, \quad \cos \beta = \frac{m_j (u_j)}{|\langle \vec{u} \rangle|}, \quad \cos \gamma = \frac{n_j (u_j)}{|\langle \vec{u} \rangle|} \]  \hspace{1cm} (6.47)

Thus, the directional permeability measured along the macroscopic flow direction \( s \) is given by

\[ \frac{1}{K_{i_n}} = \cos^2 \alpha / K_{i_1} + \cos^2 \beta / K_{i_2} + \cos^2 \gamma / K_{i_3} \]  \hspace{1cm} (6.48)

such that

\[ -\frac{\partial (p)^f}{\partial s} = \mu_T / K_{i_n} \frac{1}{|\langle \vec{u} \rangle|} \]  \hspace{1cm} (6.49)

or, in dimensionless form, as

\[ -\frac{\partial (p)^f}{\partial \eta \frac{L^2}{\mu_T |\langle \vec{u} \rangle|}} = \frac{L^2}{K_{i_n}} \]  \hspace{1cm} (6.50)

Thus, the directional permeability \( K_{i_n} \) may readily be determined by reading the intercept of the ordinate variable, as we plot \(- (\partial (p)^f / \partial s) (L^2 / \mu_T |\langle \vec{u} \rangle|) \) against \( Re_L \), as done in the study on the cross-flow case [10]. The solution surfaces of the directional permeability are constructed using the numerical values and presented in terms of \( L^2 / K_{i_n} \) against the projected angle \( \alpha' \) and the yaw angle \( \gamma \) for the cases of \( H/L = 1 \) and \( 3/2 \) in Figure 6.6(a). The solution surface changes drastically as the ratio \( H/L \) departs from unity. It is interesting to note that the effect of the projected angle \( \alpha' \) on the directional permeability is
Solution surfaces for directional permeability; (a) numerical experiments, $H/L = 1, H/L = \frac{3}{2}$, (b) correlations, $H/L = 1, H/L = \frac{3}{2}$.

**TABLE 6.1**

<table>
<thead>
<tr>
<th>$H/L$ ($\phi$)</th>
<th>$L^2/K_{f_1}$</th>
<th>$L^2/K_{f_2}$</th>
<th>$L^2/K_{f_3}$</th>
<th>$v_1L$</th>
<th>$v_2L$</th>
<th>$b_1L$</th>
<th>$b_2L$</th>
<th>$b_3L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (0.750)</td>
<td>76</td>
<td>76</td>
<td>41</td>
<td>0.2</td>
<td>0.2</td>
<td>8.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\frac{3}{2}) (0.833)</td>
<td>16</td>
<td>55</td>
<td>13</td>
<td>0.1</td>
<td>0.6</td>
<td>3.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 (0.875)</td>
<td>7</td>
<td>42</td>
<td>6</td>
<td>0.05</td>
<td>0.8</td>
<td>1.2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

totally absent for the arrangement $H/L = 1$. The coefficients $K_{f_1}, K_{f_2},$ and $K_{f_3}$ in the proposed expression (6.48) may be determined by fitting the numerical results against the solution surfaces based on Eq. (6.48). Such solution surfaces generated by the proposed Eq. (6.48) are presented in Figure 6.6(b) for comparison. The numerical values of $K_{f_1}, K_{f_2},$ and $K_{f_3}$ determined in this manner are listed in Table 6.1. The validity of the proposed Eq. (6.48) with
FIGURE 6.7
Directional permeability at $\gamma = \pi/2$.

the values listed in Table 6.1 can be examined further by plotting $L^2/K_{f_n}$ as shown in Figure 6.7 for the case of $\gamma = \pi/2$, where the fluid flows perpendicularly to the rods. It is seen that the numerical results closely follow the curves generated from Eq. (6.48).

6.9 Determination of Forchheimer Tensor

When the velocity (i.e., Reynolds number) is sufficiently high, the inertial Forchheimer term describing the form drag predominates over the Darcy term such that

$$-\frac{\partial (p)^f}{\partial x_i} = \left(\mu_f K_{ij}^{-1} + \rho_f b_{ij} \langle \vec{u} \rangle \right) \langle u_i \rangle \approx \rho_f b_{ij} \langle \vec{u} \rangle \langle u_j \rangle$$

(6.51)

Usually, the principal axes of the permeability tensor $K_{ij}^{-1}$ do not coincide with those of the inertial Forchheimer tensor $b_{ij}$. For the orthotropic media in consideration, however, the tensors $b_{ij}$ should be symmetric, and hence, they must satisfy the following symmetric conditions:

$$\frac{\partial b_{ij}}{\partial \alpha} \bigg|_{\alpha=0,\pi/2} = \frac{\partial b_{ij}}{\partial \beta} \bigg|_{\beta=0,\pi/2} = \frac{\partial b_{ij}}{\partial \gamma} \bigg|_{\gamma=0,\pi/2} = 0$$

(6.52)
where
\[ b_{ts} = b_{lj} \frac{\langle u_i \rangle \langle u_j \rangle}{|\langle \bar{u} \rangle|^2} \]  \hspace{1cm} (6.53)

is the directional Forchheimer coefficient measured along the macroscopic flow direction \( s \). One of the simplest functions that satisfy these conditions may be:
\[
b_{lj} = b_{l1} (l_i l_j) + b_{l2} (m_i m_j) + b_{l3} (n_i n_j) + bb_{l1} \cos \alpha \cos \beta (\langle l_i m_j \rangle + \langle l_j m_i \rangle) \\
+ bb_{l2} \cos \beta \cos \gamma (\langle m_i n_j \rangle + \langle m_j n_i \rangle) + bb_{l3} \cos \gamma \cos \alpha (\langle n_i l_j \rangle + \langle n_j l_i \rangle) \]  \hspace{1cm} (6.54)

which results in
\[
b_{ts} = b_{l1} \cos^2 \alpha + b_{l2} \cos^2 \beta + b_{l3} \cos^2 \gamma + 2bb_{l1} \cos^2 \alpha \cos^2 \beta \\
+ 2bb_{l2} \cos^2 \beta \cos^2 \gamma + 2bb_{l3} \cos^2 \gamma \cos^2 \alpha \]  \hspace{1cm} (6.55)

such that
\[
-\frac{\partial \langle p \rangle^f}{\partial s} = \frac{\mu_f}{K_{in}} |\langle \bar{u} \rangle| + \rho_f b_{ts} |\langle \bar{u} \rangle|^2 \]  \hspace{1cm} (6.56)

or, in dimensionless form, as
\[
-\frac{\partial \langle p \rangle^f}{\partial s} \frac{L}{\rho_f |\langle \bar{u} \rangle|^2} = \frac{L^2}{K_{in} Re_L} + b_{ts} L \]  \hspace{1cm} (6.57)

Plotting the results of macroscopic pressure gradient in terms of \(-\langle \partial \langle p \rangle^f/\partial s \rangle L/\rho_f |\langle \bar{u} \rangle|^2\) and reading the horizontal asymptotes, we can readily determine the directional Forchheimer constant.

The numerical values of the directional Forchheimer constant for the cases of \( H/L = 1 \) and \( \frac{3}{2} \) are shown in terms of the solution surfaces of \( b_{ts} L \) in Figure 6.8(a). These figures clearly show that, for fixed \( \gamma \), the directional Forchheimer constant attains its peak around \( \alpha' = \pi/2 \), while, for fixed \( \alpha' \), it decreases monotonically from \( \gamma = \pi/2 \) to 0.

From this observation, we find that the coefficients and \( \rho_f \) is nonzero while \( b_{ts}, bb_{l2}, \) and \( bb_{l3} \) in Eq. (6.55) should vanish for the bank of cylinders, such that
\[
b_{ts} = b_{l1} \cos^2 \alpha + b_{l2} \cos^2 \beta + 2bb_{l1} \cos^2 \alpha \cos^2 \beta \\
= (b_{l1} \cos^2 \alpha' + b_{l2} \sin^2 \alpha' + 2bb_{l1} \cos^2 \alpha' \sin^2 \alpha' \sin^2 \gamma) \sin^2 \gamma \]  \hspace{1cm} (6.58)
FIGURE 6.8
Solution surfaces for directional Forchheimer coefficient; (a) numerical experiments, \( H/L = 1 \), \( H/L = \frac{3}{2} \); (b) correlations, \( H/L = 1 \), \( H/L = \frac{3}{2} \).

The corresponding \( b_{\alpha}L \) surfaces based on the proposed expression (6.58) with the values of \( b_{H1} \), \( b_{f1} \), and \( bb_{f1} \) as listed in Table 6.1 are presented in Figure 6.8(b) for comparison. Furthermore, the numerical results of the directional Forchheimer constant obtained with \( \gamma = \pi/2 \) for \( H/L = 1 \), \( \frac{3}{2} \), and 2 are presented in Figure 6.9 as a function of the cross-flow angle \( \alpha (=\alpha') \). In the same figure, the solid curves generated from the proposed Eq. (6.58) are presented to elucidate the validity of the proposed expression. Note that, for this case of \( \gamma = \pi/2 \), the foregoing equation reduces to

\[
b_{\alpha} = b_{H1} \cos^2 \alpha + b_{f1} \sin^2 \alpha + 2bb_{f1} \cos \alpha \sin^2 \alpha
\]

(6.59)

It is interesting to note that the numerical results for the cases \( H/L = \frac{3}{2} \) and 2 show two consecutive peaks, while the model Eq. (6.59) yields only one
peak (the first peak). The second peak appears when the macroscopic flow angle, $\alpha$, reaches roughly $\tan^{-1}(H/L)$. Note that, for the case of $H/L = 1$, this second peak coincides with the first one. Unfortunately, the model equation is incapable of describing the second peak.

Zukauskas [24] assembled the experimental data for the fully developed pressure drop across the tube banks in both inline-square and staggered-triangle arrangements, and presented a chart for the Euler number (i.e., the dimensionless macroscopic pressure drop over a unit). His inline-square arrangement corresponds to the present arrangement with $\alpha = 0$, $\gamma = \pi/2$, and $L/D = 2$. However, it is noted that, in reality, the macroscopic flow direction rarely coincides with the principal axes, since even small disturbances at a sufficiently high Reynolds number deviate the flow from the axis. Thus, it is understood that the chart provided by Zukauskas gives only the average level of the pressure drop within a range of small $\alpha$ (say $0^\circ < \alpha < 5^\circ$). The dimensionless macroscopic pressure gradient $-(\partial (p)/\partial s)/(L/\rho \langle \bar{u} \rangle^2)$ for the case of $\gamma = \pi/2$ and $L/D = 2$ is plotted against $Re_L$, in Figure 6.10, where the curves generated from the model Eq. (6.57) with the numerical values taken from Table 6.1 and Figure 6.9 (note that $b_{fL}L = 0.2$ and 0.6, for $\alpha = 0^\circ$ and $5^\circ$, respectively) are drawn together with the empirical chart provided by Zukauskas for the inline-square arrangement. The agreement between these curves appears fairly good.
6.10 Determination of Interfacial Heat Transfer Coefficient

The interfacial heat transfer coefficient as defined by Eq. (6.6) may be obtained by substituting the microscopic temperature results into the following equation:

\[
h_i = \frac{1}{(V)} \int_{A_{\text{int}}} k_i \nabla T \cdot d\vec{A} = \frac{1}{A_{\text{fluid}}} \int_{A_{\text{int}}} \left( -k_i (\partial T / \partial n) \right) dP
\]

where \( A_{\text{int}} \) is the total interface between the fluid and solid, while \( d\vec{A} \) is its vector element pointing outward from the fluid to solid side. In Figure 6.11, the heat transfer results obtained at \( \alpha = 0 \) and \( \pi / 4 \) for the cross-flows (i.e., \( \gamma = \pi / 2 \)) are presented in terms of the interfacial Nusselt number \( N_{\text{uL}} = h_i L / k_i \) against the Reynolds number \( Re_L \). The figure suggests that the lower and higher Reynolds number data follow two distinct limiting lines for the case of nonzero \( \alpha \), namely, \( \alpha = \pi / 4 \). The lower Reynolds number data stay constant for the given array and flow angle, whereas the high Reynolds number data vary in proportion to \( Re_L^{0.6} \).

Another series of computations changing the Prandtl number, conducted following Kuwahara et al. [19], revealed that the exponents associated with the Reynolds and Prandtl numbers are the same as those Wakao and Kagoue [25] observed as collecting and scrutinizing reliable experimental data on interfacial convective heat transfer coefficients in packed beds. The similarity, albeit the difference in the Reynolds number dependence, between the Nusselt number \( N_{\text{uL}} \) and the macroscopic pressure gradient as given by Eq. (6.56) is noteworthy, which prompts us to model the directional Nusselt
number as follows:

\[
Nu_L = \frac{h_t L}{k_t} = c_t + d_t Re_L^{0.6} Pr_t^{1/3}
\]  \hspace{1cm} (6.61)

In the figure, the experimental correlation proposed by Zukauskas [26] for the heat transfer from the circular tubes in staggered banks is compared with the present results obtained for the case of \( \alpha = \pi/4 \), \( \gamma = \pi/2 \), and \( H/L = 1 \). (Note \( Nu_t \approx Nu_{t,1/2} \) and \( Re_t \approx Re_L \) in Eq. (6.39) of Zukauskas since \( D/L = \frac{1}{2} \).) The present results follow closely along the experimental correlation of Zukauskas as increasing the Reynolds number. Grimison [27] carried out an exhaustive experiment to investigate heat transfer from tube rows of a bank in both staggered and aligned arrangements with respect to the direction of the macroscopic flow. His case, in which the ratio of the transverse pitch to tube diameter and that of the longitudinal pitch to tube diameter are 3 and 1.5, respectively, gives a configuration close to the present orthogonal configuration with \( \alpha = \pi/4 \), \( \gamma = \pi/2 \), and \( H/L = 1 \). Thus, the experimental correlation established by Grimison for the case is also presented in the figure, which agrees very well with the present numerical results. These correlations are believed to hold for a comparatively wide Reynolds number range, covering from a predominantly laminar flow regime to turbulent flow regime.

Following the procedure similar to the one adopted for determining the directional permeability, the coefficient \( c_t = \frac{Nu_{t,1/2}Re_t}{Re_L \to 0} \) for each macroscopic flow angle is evaluated and plotted in terms of the solution surfaces in Figure 6.12(a), using the low Reynolds number data. It is noted that the effect of the projected angle \( \alpha' \) on the interfacial heat transfer coefficient is totally absent for the arrangement \( H/L = 1 \).
FIGURE 6.12
Solution surfaces for directional heat transfer coefficient at small Reynolds number; (a) numerical experiments, $H/L = 1$, $H/L = \frac{3}{2}$, (b) correlations, $H/L = 1$, $H/L = \frac{3}{2}$.

The similarity between the solution surfaces of $c_f$ and those of $L^2/K_{tn}$ is obvious, which leads us to introduce a functional form as follows:

$$c_f = \left( \frac{n_c}{c_{f_1} \cos^2 \alpha} + \frac{n_c}{c_{f_2} \cos^2 \beta} + \frac{n_c}{c_{f_3} \cos^2 \gamma} \right)^{1/n_c} \quad (6.62)$$

such that $c_f$ reduces to $c_{f_1}$, $c_{f_2}$, and $c_{f_3}$ for $\alpha = 0$, $\beta = 0$, and $\gamma = 0$, respectively, as it should.

Careful examination of the numerical results over the whole domain within $0 \leq \alpha' \leq \pi/2$ and $0 \leq \gamma \leq \pi/2$ suggests that $n_c$ is close to minus one, which leads us to a harmonic mean expression as

$$\frac{1}{c_f} = \frac{\cos^2 \alpha}{c_{f_1}} + \frac{\cos^2 \beta}{c_{f_2}} + \frac{\cos^2 \gamma}{c_{f_3}} \quad (6.63)$$
TABLE 6.2

Coefficients for Directional Nusselt Number

<table>
<thead>
<tr>
<th>$H/L$ ($\phi$)</th>
<th>$c_{f1}$</th>
<th>$c_{f2}$</th>
<th>$c_{f3}$</th>
<th>$n_c$</th>
<th>$d_{t1} = d_{t2}$</th>
<th>$n_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (0.750)</td>
<td>11</td>
<td>11</td>
<td>8.6</td>
<td>-1.0</td>
<td>0.90</td>
<td>4.5</td>
</tr>
<tr>
<td>$\frac{2}{3}$ (0.833)</td>
<td>4.8</td>
<td>14</td>
<td>5.2</td>
<td>-1.0</td>
<td>0.77</td>
<td>4.3</td>
</tr>
<tr>
<td>2 (0.875)</td>
<td>3.2</td>
<td>16</td>
<td>3.6</td>
<td>-1.0</td>
<td>0.67</td>
<td>4.5</td>
</tr>
</tbody>
</table>

FIGURE 6.13

Effect of the cross-flow angle $\alpha$ on the coefficient $c_f$ at $\gamma = \pi/2$.

The values of $c_{f1}$, $c_{f2}$, and $c_{f3}$ listed in Table 6.2 have been determined by fitting the numerical results against the foregoing equation. The resulting surfaces based on the proposed expression (6.63) are presented in Figure 6.12(b) for their comparison with the surfaces based on the numerical experiments shown in Figure 6.12(a). Furthermore, Figure 6.13 shows the numerical results of $c_f$ obtained at $\gamma = \pi/2$ for the three distinct arrangements, namely, $H/L = 1$, $\frac{2}{3}$, and 2. The solid curves in the figure are generated from the proposed Eq. (6.63) with the values of $c_{f1}$ and $c_{f2}$ as listed in Table 6.2.

The second coefficient $d_t$ may be determined using the data $\frac{Nu_L}{Re_s^{0.6} Pr_s^{1/3}}$ in the high Reynolds number range. The resulting solution surfaces of $d_t$ are presented in Figure 6.14 for $H/L = 1$ and $\frac{2}{3}$. Unlike the Forchheimer coefficient $b_{in}$, the coefficient $d_t$ stays roughly constant for a fixed yaw angle $\gamma$.

More careful observation on the solution surfaces reveals that the coefficient $d_t$ drops abruptly as the projected angle $\alpha'$ reaches close to either 0 or $\pi/2$ (in which the fluid flows along the principal axis of the structure). However, as already pointed out, it is quite unlikely to have the macroscopic flow align perfectly with the principal axes. Thus, we may assume that $d_t$ is the function
FIGURE 6.14
Solution surfaces for directional heat transfer coefficient at large Reynolds number; (a) numerical experiments, \( H/L = 1, H/L = \frac{3}{2} \), (b) correlations, \( H/L = 1, H/L = \frac{3}{2} \).

of the yaw angle \( \gamma \) alone, namely, \( d_f = d_f(\gamma) \). It is interesting to note that \( d_f = d_f(\gamma') \) is consistent with the idea of the effective velocity \( u_{eff} = |\langle \bar{u} \rangle| \sin \gamma \) used in the hot-wire anemometry. Thus, we may model \( d_f \) as

\[
d_f = \left( d_{f1}^{nd} \sin^2 \gamma + d_{f2}^{nd} \cos^2 \gamma \right)^{1/n_d} \tag{6.64}
\]

A careful observation on the solution surfaces leads us to \( d_{f3} \approx 0 \), and also reveals the values of \( d_{f1} \) and \( n_d \) as listed in Table 6.2. Thus, we propose the expression as follows:

\[
Nu_L = \left( c_{f1}^{nc} \cos^2 \alpha + c_{f2}^{nc} \cos^2 \beta + c_{f3}^{nc} \cos^2 \gamma \right)^{1/nc} + d_{f1} \sin^{2/n_d} \gamma Re_L^{0.6} Pr_f^{1/3} \tag{6.65a}
\]
or

$$Nu_D = \frac{1}{2} \left( c_1^\alpha \cos^2 \alpha + c_2^\beta \cos^2 \beta + c_3^\gamma \cos^2 \gamma \right)^{1/n_c} + \frac{d_1}{20.4} \sin^{2/n_d} \gamma Re_D^{0.6} Pr_f^{1/3} \quad (6.65b)$$

Note that the exponents $n_c = -1$ and $n_d = \frac{9}{2}$ irrespectively of the value of $H/L$, while the coefficients $c_1, c_2, c_3$, and $d_1$ depend on that particular geometrical configuration.

Zukauskas [24] investigated the effect of the yaw angle on the interfacial heat transfer rate. He varied the yaw angle $\gamma$ for both staggered and aligned arrangements, and compared the corresponding heat transfer rates for the same Reynolds number. He pointed out that the data when normalized by the value obtained at $\gamma = \pi/2$ for all staggered and inline arrangements, namely $Nu_D/Nu_D|\gamma=\pi/2$, can be approximated by a single curve irrespective of the Reynolds number. His data for both staggered and inline arrangements are plotted in Figure 6.15 together with the expression based on the model Eq. (6.65b), namely,

$$\frac{Nu_D}{Nu_D|\gamma=\pi/2} \approx \sin^{2/n_d} \gamma = \sin^{4/9} \gamma \quad (6.66)$$

for the case of sufficiently high Reynolds number. The agreement between the experimental data and the curve based on Eq. (6.66) is fairly good, which indicates the validity of the model Eq. (6.65b). It should also be noted that the staggered arrangement corresponds to the case of $\alpha' = \pi/4$ while the inline arrangement to the case in which $\alpha'$ is close to zero (but $\alpha' \neq 0$ since the macroscopic flow direction never coincides with the principal axis of the
structure). Thus, these experimental data substantiates our finding based on the numerical experiment, namely, that the multiplicative constant for the interfacial Nusselt number $d_f$ stays virtually constant (irrespective of $\alpha'$) for a fixed yaw angle, as illustrated by the solution surfaces in Figure 6.14.

### 6.11 Conclusions

A numerical modeling strategy for dealing with three-dimensional flow and heat transfer within highly anisotropic porous media has been proposed to attack complex fluid flow and heat transfer associated with heat transfer equipment. An appropriate set of the periodic boundary conditions has been derived appealing to the concept of VAT, and applying it to a macroscopically uniform flow through an isothermal porous medium of infinite extent. For three-dimensional heat and fluid flow through a two-dimensional structure, a quasi-three-dimensional calculation procedure is found possible. The procedure can be exploited to investigate three-dimensional heat and fluid flow through a bank of cylinders in yaw, which represents a numerical model for manmade structures such as plate-fin heat exchangers. Only one structural unit was taken as a calculation domain, noting the periodicity of the structure. This inexpensive and yet efficient numerical calculation procedure based on one structural unit along with periodic boundary conditions was employed to conduct extensive three-dimensional calculations for a number of sets of the porosity, degree of anisotropy, Reynolds number, Prandtl number, and macroscopic flow direction. The numerical results, thus obtained at the pore level, were integrated over a structural unit to determine the permeability tensor, Forchheimer tensor, and interfacial heat transfer coefficient, so as to elucidate the effects of yaw angle on these macroscopic flow and heat transfer characteristics. Upon examining these numerical experimental data, a useful set of explicit expressions for the permeability tensor, Forchheimer tensor, and interfacial heat transfer coefficient have been established for the first time, such that one can easily evaluate the pressure drop and heat transfer rate from the bank of cylinders in yaw. The systematic modeling procedure proposed in this study can be utilized to conduct subscale modeling of manmade structures needed in the possible applications of a VAT to investigate flow and heat transfer within complex heat and fluid flow equipment consisting of small elements.

### Nomenclature

- $\bar{A}$: surface area vector
- $A_{\text{int}}$: total interface between the fluid and solid
Forchheimer coefficient tensor, directional Forchheimer coefficient
specific heat capacity at constant pressure
coefficients associated with directional Nusselt number
size of square rod
size of structural unit
interfacial convective heat transfer coefficient
thermal conductivity
permeability tensor, directional permeability
Prandtl number
microscopic velocity components in the $x$, $y$, and $z$ directions
microscopic temperature
microscopic pressure
Reynolds number based on $L$ and the Darcian velocity
Reynolds number based on $D$ and the Darcian velocity
elementary representative volume
Cartesian coordinates
angles between the macroscopic velocity vector and principal axes
projected angle, cross-flow angle
kinematic viscosity
density
viscosity
porosity

Subscripts and superscripts

<table>
<thead>
<tr>
<th>Subscript/superscript</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$f$</td>
<td>fluid</td>
</tr>
<tr>
<td>$s$</td>
<td>solid</td>
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Special symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\langle \rangle$</td>
<td>volume-average</td>
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<tr>
<td>$\langle \rangle_f$</td>
<td>intrinsic average</td>
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</tbody>
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References


